Math Meth Oper Res (2000) 51 : 69–89

Mathematical Methods
of Operations Research

A flexible approach to location problems

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Abstract. When dealing with location problems we are usually given a set of existing facilities and we are looking for the location of one or several new facilities. In the classical approaches weights are assigned to existing facilities expressing the importance of the new facilities for the existing ones.

In this paper, we consider a pointwise defined objective function where the weights are assigned to the existing facilities depending on the location of the new facility. This approach is shown to be a generalization of the median, center and centdian objective functions. In addition, this approach allows the formulation of completely new location models. Efficient algorithms as well as structural results for this algebraic approach to location problems are presented. A complexity analysis and extensions to the multifacility and restricted case are also considered.

Key words: Location theory, global optimization, algebraic optimization, convexity

1 Introduction

In the last three decades much research has been done in the field of continuous location theory and very many different models have been developed. For a comprehensive overview the reader is referred to Plastria's chapter in the book of Drezner [18].

In the following we will introduce a new model for location problems. This new model provides a common framework for the classical continuous location problems and allows an algebraic approach to these problems. Moreover, this flexible approach leads to completely new objective functions for location

Manuscript received: May 1999/final version received: July 1999

problems. It is also worth noting that the approach presented in the following emphasizes the role of the clients seen as a collective. From this point of view the quality of the service provided by a new facility to be located does not depend on the specific names given to the demand points by the modeler. Different ways to assign names to the demand points should not change the quality of the service, i.e. a symmetry principle must hold. This principle being important in location problems is inherent to this model. In fact, this model is much more than a simple generalization of some classical models in locational analysis because any location problem whose objective function is a monotone, symmetrical norm of the vector of distances reduces to it [19]. After this short characterization of the new model we will introduce it formally.

We are given a gauge $\gamma(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ to measure distances, a set of demand points $A = \{a_1, a_2, \ldots, a_M\} \subset \mathbb{R}^2$ (representing existing facilities or clients) and two sets of non negative scalars $W = \{w_1, \ldots, w_M\}$ and $\Lambda = \{\lambda_1, \ldots, \lambda_M\}$. The element $w_i \in W$ is the weight of importance given to the existing facility a_i, and the elements of Λ allow us to choose between different kinds of objective functions.

Given a permutation σ of the set $\{1, \ldots, M\}$ satisfying

$$w_{\sigma_1}\gamma(x - a_{\sigma_1}) \leq w_{\sigma_2}\gamma(x - a_{\sigma_2}) \leq \cdots \leq w_{\sigma_M}\gamma(x - a_{\sigma_M})$$

we denote $\gamma(x - A)_{(i)} = w_{\sigma_i}\gamma(x - a_{\sigma_i})$.

The ordered Weber problem is then given by:

$$\min_{x \in \mathbb{R}^2} F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x - A)_{(i)}. \tag{1}$$

Note that the problem is well-defined even if ties occur. In that case any order of the tied positions gives the same value.

Theoretical properties of (1) have been studied in a different setting in [19]. To describe different types of location problems we use a 5-position classification scheme Pos1/Pos2/Pos3/Pos4/Pos5, which allows us to indicate the number of new facilities (Pos1), the type of the problem as planar, network-based, discrete, etc. (Pos2), any assumption and restriction such as $w_m = 1$ for all $m \in \mathcal{M}$, etc. (Pos3), the type of distance function such as l_p, general distance function d, etc. (Pos4), and the type of objective function (Pos5) (see [16] for further details). According to this scheme we will refer to the problem described in (1) as $1/\mathbb{R}^2/\bullet/\gamma_B/\Sigma_{ord}$.

The reader may note that problem $1/\mathbb{R}^2/\bullet/\gamma_B/\Sigma_{ord}$ is somehow similar to the well-known Weber Problem, but it is more general because it includes as particular instances the Weber problem ($\lambda_1 = \lambda_2 = \cdots = \lambda_M = 1$), the α-centdian problem ($\lambda_1 = \cdots \lambda_{M-1} = 1 - \alpha$ and $\lambda_M = 1$) and the center problem ($\lambda_1 = \cdots = \lambda_{M-1} = 0$ and $\lambda_M = 1$).

But, as already mentioned at the beginning of the introduction, also new useful objective functions can easily be modeled. Assume, for example, that we are not only interested in minimizing the distance to the most remote client (center objective function), but instead we would like to minimize the average distance to the 5 most remote clients (or any other number). This can easily be modeled by setting $\lambda_{M-4}, \ldots, \lambda_M = 1$ and all other lambdas to zero. This

k-centra problem is a different way of combining average and worst-case behavior.

Also ideas from robust statistics can be implemented by only taking into account always the k_1 nearest and simultaneously the k_2 farthest. Of course, we could also just exclude always the k_1 nearest and simultaneously the k_2 farthest. Moreover, the direct relation of the lambdas to the choice of the objective function will be quite useful for scenario analysis.

Example 1.1. Consider three demand points $a_1 = (1,2)$, $a_2 = (3,5)$ and $a_3 = (2,2)$ with weights $w_1 = w_2 = w_3 = 1$. Now choose $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then we get $F(x) = \sum_{i=1}^{3} \|x - a_i\|$, i.e. the Weber problem. For the second case choose $\lambda_1 = \lambda_2 = 1/2$ and $\lambda_3 = 1$ then we get: $F(x) = 1/2\sum_{i=1}^{3} \|x - a_i\| + 1/2 \max_{1 \le i \le 3}\|x - a_i\|$, i.e. the 1/2-centdian problem. Finally choose $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$ and we get: $F(x) = \max_{1 \le i \le 3}\|x - a_i\|$, i.e. the center problem.

Also note that the objective function of this problem is region-wise defined and in general non-convex if no additional assumptions are made on the set Λ (see [6] for further details).

Example 1.2. Consider two demand points $a_1 = (0,0)$ and $a_2 = (10,5)$, $\lambda_1 = 100$ and $\lambda_2 = 1$ with l_1-norm and $w_1 = w_2 = 1$. We obtain only two optimal solutions to Problem (1), lying in each demand point. Therefore the objective function is not convex since we have a non-convex optimal solution set.

$$F(a_1) = 100 \times 0 + 1 \times 15 = 15$$

$$F(a_2) = 100 \times 0 + 1 \times 15 = 15$$

$$F(\tfrac{1}{2}(a_1 + a_2)) = 100 \times 7.5 + 1 \times 7.5 = 757.5$$

See Figure 1.

These two characteristics allow to model many different problems as we will show in the following.

The aforementioned paper by Puerto and Fernández [19] focuses only on developing the theoretical properties of this problem. Neither algorithms have been presented nor complexity aspects have been addressed. Exactly this will be the aim of this paper.

The outline of the paper is as follows: first geometrical properties of (1) with polyhedral gauges are exploited. Then an efficient algorithm for the single facility case is given. The next section is devoted to extensions of Problem (1) to the multifacility case. After that, the cases of restricted problems and general gauges are investigated and an approximation result is given. The paper ends with some conclusions and an outlook to future research. In order to keep the presentation of this paper as clear as possible, we discuss all results for the planar case only. Nevertheless most of the following principles can directly be adapted to the n-dimensional case.

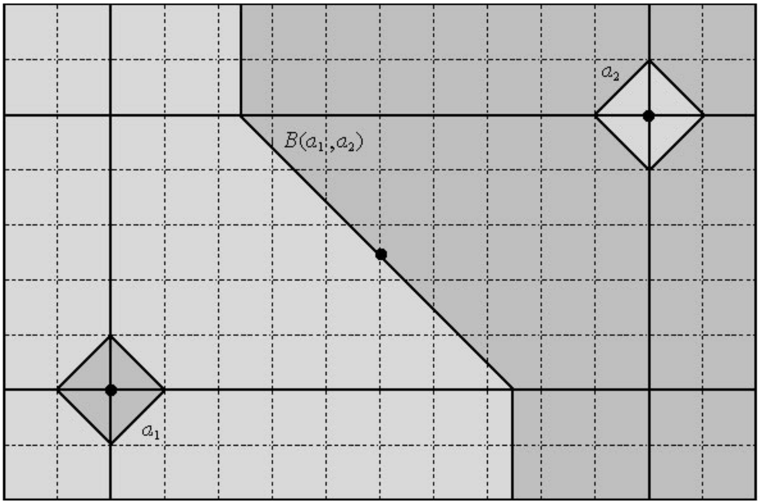

Fig. 1. Illustration to Example 1.2

2 Geometrical properties

We are mainly interested in problems with polyhedral gauges. For this reason we will assume in the rest of the paper that $B \subseteq \mathbb{R}^2$ is a bounded polytope whose interior contains the zero and we denote the set of extreme points of B by $Ext(B) = \{e_g : g = 1, \ldots, G\}$.

The polar set B^o of B is given by

$$B^o = \{x \in \mathbb{R}^2 : \langle x, p \rangle \leq 1 \quad \forall p \in B\}.$$

In the polyhedral case, B^o is also a polytope, whose extreme points are $\{e_g^o : g = 1, 2, \ldots, G\}$, in $\mathbb{R}^2$, see [21] and [8].

The normal cone to B at x is given by

$$N(B, x) := \{p \in \mathbb{R}^2 : \langle p, y - x \rangle \leq 0 \quad \forall y \in B\} \tag{2}$$

and the boundary of B is denoted by $bd(B)$.

In this section we address properties of the planar formulation of Problem (1) (denoted by $1/\mathbb{R}^2/\bullet/\gamma_B/\Sigma_{ord}$) which give us specific insights into the geometry of the considered model. We define fundamental directions $d_1, \ldots, d_G$ as the halflines defined by 0 and $e_1, \ldots, e_G$. Further, we define Γ_g as the cone generated by d_g and d_{g+1} (fundamental directions of B) where $d_{G+1} := d_1$. Let $\pi = (p_i)_{i \in \mathcal{M}}$ be a family of elements of $\mathbb{R}^2$ such that $p_i \in B^o$ for each $i \in \mathcal{M}$ and let $C_\pi = \bigcap_{i \in \mathcal{M}} (a_i + N(B^o, p_i))$. A nonempty convex set C is called an elementary convex set (e.c.s.) if there exists a family π such that $C_\pi = C$.

We can obtain the elementary convex sets also as intersections of cones generated by fundamental directions of these balls pointed at each demand point. Therefore each elementary convex set is a polyhedron whose vertices are called intersection points (see Figure 1). An upper bound of the number of elementary convex sets is $O(M^2G^2)$. For further details see Durier and Michelot [8].

As we have already seen in the last section we do not have a unified linear representation of the objective function of (1) on the whole space.

It is easy to see, that the representation may change when $\gamma(x - a_i) - \gamma(x - a_j)$ becomes 0 for some $i, j \in \{1, \dots, M\}$ with $i \neq j$. We will develop in the following a geometrical description of the regions where the representation of the objective function as a weighted sum stays unchanged.

Definition 2.1. *The set $B_\gamma(a_i, a_j)$ consisting of points $\{x : w_i\gamma(x - a_i) = w_j\gamma(x - a_j), i \neq j\}$ is called bisector of a_i and a_j with respect to γ.*

As an illustration of Definition 2.1 one can see in Figure 1 the bisector line for the points a_1 and a_2 with the rectangular norm.

Proposition 2.1. *The bisector of a_i and a_j is a set of points verifying a linear equation within each elementary convex set.*

Proof: In an elementary convex set $\gamma(x - a_i)$ and $\gamma(x - a_j)$ can be written as $l_i(x - a_i)$ and $l_j(x - a_j)$ respectively, where l_i and l_j are linear functions. Therefore, $w_i\gamma(x - a_i) = w_j\gamma(x - a_j)$ is equivalent to $w_il_i(x - a_i) = w_jl_j(x - a_j)$ and the result follows. □

We will now give a more exact description of the complexity of a bisector.

Proposition 2.2. *The bisector of a_i and a_j with respect to a polyhedral gauge γ with G extreme points has at most $O(G)$ different subsets defined by different linear equations.*

Proof: By Proposition 2.1 bisectors are set of points given by linear equations within each elementary convex set. Therefore, the unique possible breakpoints may occur on the fundamental directions.

Let us denote by $L^g_{a_i}$ the fundamental direction starting at a_i with direction e_g. On this halfline the function $w_i\gamma(x - a_i)$ is linear with constant slope and $w_j\gamma(x - a_j)$ is piecewise linear and convex. Therefore, the maximum number of zeros of $w_i\gamma(x - a_i) - w_j\gamma(x - a_j)$ when $x \in L^g_{a_i}$ is two. Hence, there are at most two breakpoints of the bisector of a_i and a_j on $L^g_{a_i}$.

Repeating this argument for any fundamental direction we obtain that an upper bound for the number of breakpoints is $4G$. □

This result implies that the number of different linear expressions defining any bisector is also linear in G, the number of fundamental directions. Remark, that in some settings bisectors may have non empty interior, see for instance Figure 2, where we show the bisector set defined by the points $(0, 0)$ and $(4, 0)$ with the Tchebychev norm.

When at least two points are considered simultaneously the set of bisectors builds a subdivision of the plane (very similar to the well-known k-order Voronoi diagrams, see the book of Okabe et al. [17]). The cells of this subdivision will be called from now on ordered regions. We formally introduce this concept.

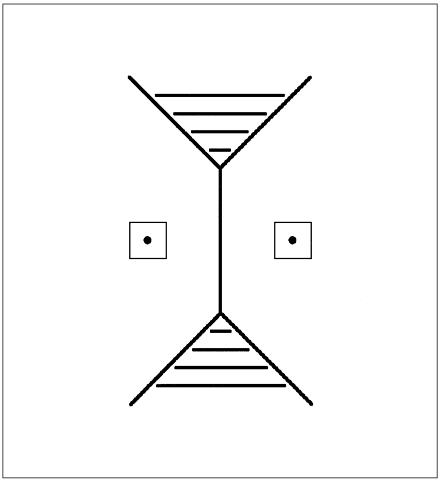

Fig. 2. An example for a degenerated bisector

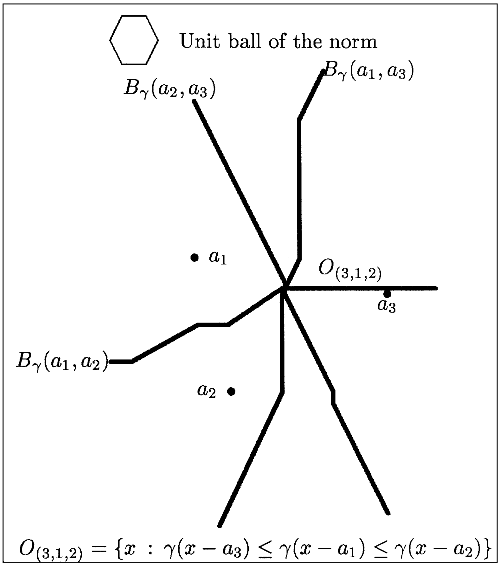

Fig. 3. Ordered regions

Definition 2.2. *Given a permutation σ on the set $\{1, 2, \ldots, M\}$ the ordered region O_σ consists of the following set of points*

$$O_\sigma = \{x \in \mathbb{R}^2 : w_{\sigma_1}\gamma(x - a_{\sigma_1}) \le \cdots \le w_{\sigma_M}\gamma(x - a_{\sigma_M})\}$$

Notice that these regions need not be convex sets, see Figure 3.

The importance of these regions is that in their intersection with any elementary convex set the problem $1/\mathbb{R}^2/\bullet/\gamma_B/\Sigma_{ord}$ behaves like a Weber problem, i.e. the objective function has a unique linear representation. The intersections between ordered regions and e.c.s. are called, according to Puerto and Fernández [19], generalized elementary convex sets (g.e.c.s.). The ordered regions play a very important role in the algorithmic approach developed for solving the problem. In terms of bisectors, these regions are cells defined by at most $M - 1$ bisectors of the set A.

However, the main disadvantage of dealing with these regions is their complexity. A naive analysis could lead to conclude that their number is $M!$ which would make the problem intractable. Fortunately, we can obtain a polynomial bound which allows us to develop in the next section an efficient algorithm for solving Problem (1).

Theorem 2.1. *An upper bound on the number of ordered regions is* $O(M^4G^2)$.

Proof: Given two bisectors with $O(G)$ linear pieces, the maximum number of intersections is $O(G^2)$. The number of bisectors of M points is $\binom{M}{2}$, so, the maximum number of intersections between them is $O\left(G^2\binom{\binom{M}{2}}{2}\right)$. By the Euler formula, the number of intersections has the same complexity as the number of regions. Hence, an upper bound for the number of ordered regions is $O(M^4G^2)$. □

A detailed analysis of this theorem shows that this bound is not too bad. Although, it is of order M^4G^2, it should be noted that the number of bisectors among the points in A is $\binom{M}{2}$ which is order M^2. Therefore, even in the most favorable case of straight lines, the number of regions in worst case analysis gives $O\left(\binom{M}{2}^2\right)$ which is, in fact $O(M^4)$. Since our bisectors are polygonal with G pieces, this bound is rather tight.

Example 2.1. Figure 3 shows the ordered regions between the points $a_1 = (0, 11)$, $a_2 = (3, 0)$ and $a_3 = (16, 8)$ with the hexagonal norm whose set of extreme points is $Ext(B) = \{(2, 0), (1, 2), (-1, 2), (-2, 0), (-1, -2), (1, -2)\}$. For instance, the region $O_{(3,1,2)}$ is the set of points

$$\{x \in \mathbb{R}^2 : \gamma(x - a_3) \leq \gamma(x - a_1) \leq \gamma(x - a_2)\}$$

Finally, we quote for the sake of completeness a result stated in [19] which geometrically characterizes the solution set of the ordered Weber location problem: "The whole set of optimal solutions of Problem (1) always coincides with some generalized elementary convex sets". This is to say, the solution set coincides with the intersection of ordered regions with elementary convex sets [8].

3 Single facility model

Having introduced the main geometrical properties of this new model, we next want to develop an algorithmic approach for solving the single facility ordered Weber problem.

It should be noted that for the Weber problem with polyhedral norms several algorithms have been proposed, see e.g. [5, 20, 21].

We start with a well-known reformulation for $F(x)$ the proof of which can be found for example in Theorem 368 in [12].

Lemma 3.1. *If the scalars in the set Λ satisfy $\lambda_1 \leq \cdots \leq \lambda_M$ then*

$$F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x - A)_{(i)} = \max_{\sigma \in P(M)} \sum_{i=1}^{M} \lambda_i w_{\sigma_i} \gamma(x - a_{\sigma_i})$$

where $P(M)$ stands for the set of permutations of $\{1, \ldots, M\}$.

This formulation can be interpreted as a worst-case approach with respect to all the possible weight arguments. First, we will consider the case where the lambdas satisfy $\lambda_1 \leq \cdots \leq \lambda_M$, since this makes the objective function much easier to handle and still includes all classical cases (center, median, centdian).

Lemma 3.2. *F is a convex function.*

Proof: By the previous lemma, $F(x)$ is the maximum of convex functions and is therefore convex. □

Moreover, Puerto and Fernández [19] proved that the set of optimal solutions of Problem (1) always coincides with some generalized elementary convex sets. However, the large number of generalized elementary convex sets requires some kind of good enumeration scheme to derive an algorithm.

Since we restrict ourselves to polyhedral gauges a simple approach can be given. Within an ordered region O_σ, consider the following linear program:

$$\begin{aligned} \min \quad & \sum_{i=1}^{M} \lambda_i z_{\sigma_i} \\ s.t. \quad & w_i \langle e_g^0, x - a_i \rangle \leq z_i \quad e_g^0 \in ext(B^0), \quad i = 1, 2, \ldots, M \\ & z_{\sigma_i} \leq z_{\sigma_{i+1}} \qquad\qquad i = 1, 2, \ldots, M - 1. \end{aligned} \tag{P_σ}$$

Lemma 3.3. *For any $x \in O_\sigma$ we have:*

1. *$x = (x_1, \ldots, x_n)$ can be extended to a feasible solution $(x_1, \ldots, x_n, z_1, \ldots, z_M)$ of P_σ.*
2. *The optimal objective value of P_σ with x fixed equals $F(x)$.*

Proof: ad 1) Since $\gamma(x - a_i) = \max_{e_g^0 \in ext(B^0)} \langle e_g^0, x - a_i \rangle$ (see [21]), we can set $z_i := w_i \gamma(x - a_i)$ which satisfies the first set of inequalities in P_σ. From $x \in O_\sigma$ we have $w_{\sigma_i} \gamma(x - a_{\sigma_i}) \leq w_{\sigma_{i+1}} \gamma(x - a_{\sigma_{i+1}})$ for $i = 1, \ldots M$. Therefore $z_{\sigma_i} \leq z_{\sigma_{i+1}}$ and also the second set of inequalities of P_σ is fulfilled.

ad 2) Since by the first set of inequalities in P_σ we always have $z_i \geq w_i \max_{e_g^0 \in ext(B^0)} \langle e_g^0, x - a_i \rangle$. Therefore in an optimal solution (x, z^*) of P_σ for fixed x we get

$$z_i^* = w_i \max_{e_g^0 \in ext(B^0)} \langle e_g^0, x - a_i \rangle = w_i \gamma(x - a_i).$$

This means for the objective function value of P_σ

$$\sum_{i=1}^{M} \lambda_i z_{\sigma_i}^* = \sum_{i=1}^{M} \lambda_i w_{\sigma_i} \gamma(x - a_{\sigma_i}) = F(x). \qquad \square$$

Corollary 3.1. *If for an optimal solution (x^*, z^*) of P_σ we have $x^* \in O_\sigma$ then x^* is also an optimal solution to the ordered Weber problem restricted to O_σ.*

Lemma 3.4. *If for an optimal solution (x^*, z^*) of P_σ we have $x^* \in O_{\sigma'}$ and $x^* \notin O_\sigma$ with $\sigma \neq \sigma'$ then*

$$\min_{x \in O_{\sigma'}} F(x) < \min_{x \in O_\sigma} F(x)$$

Proof: Since $\sigma \neq \sigma'$ there exist at least two indices i and j such that for $x \in O_\sigma$ we have $w_i \gamma(x - a_i) \leq w_j \gamma(x - a_j)$ and for $x \in O_{\sigma'}$ we have $w_i \gamma(x - a_i) > w_j \gamma(x - a_j)$. But (x^*, z^*) is feasible for P_σ, which means $z_i^* \leq z_j^*$ and

$$z_i^* \geq w_i \max_{e_g^0 \in ext(B^0)} \langle e_g^0, x^* - a_i \rangle = w_i \gamma(x^* - a_i),$$

$$z_j^* \geq w_i \max_{e_g^0 \in ext(B^0)} \langle e_g^0, x^* - a_i \rangle = w_i \gamma(x^* - a_i).$$

Together we get for $x^* \in O_{\sigma'}$ and (x^*, z^*) being feasible for P_σ that $z_j^* > w_j \gamma(x^* - a_j)$ in P_σ. This implies that the optimal objective value for P_σ which is $\sum_{i=1}^{M} \lambda_i z_{\sigma_i}^* > F(x^*)$. But from Lemma 3.3 we know that since $x^* \in O_{\sigma'}$ the optimal objective value of $P_{\sigma'}$ equals $F(x^*)$. $\square$

Based on Lemma 3.4 and the fact that the objective function is globally convex we develop a descent algorithm for this problem. For each ordered region we solve the problem as a linear program which geometrically means either finding the locally best solution in this ordered region or detecting that this region does not contain the global optimum by Lemma 3.4. In the former case two situations may occur. First, if the solution lies in the interior of the considered region (in $\mathbb{R}^2$) then by convexity this is the global optimum and secondly, if the solution is on the boundary we have to do a local search in the neighborhood regions where this point belongs to. This is done in Step 7 of the algorithm. It is worth noting that to accomplish this search a list $\mathscr{L}$ containing the already visited neighborhood regions is used in the algorithm. Besides, it is also important to realize that neither Step 2 nor Step 5 need to explicitly construct the corresponding ordered region. It suffices to evaluate and to sort the distances to the demand points.

ALGORITHM 3.1.

Solving $1/\mathbb{R}^2/\bullet/\gamma_B/\sum_{ord}$

Step 1 *Choose* x^o *as an appropriate starting point. Initialize* $\mathcal{L} := \emptyset$, $y^* = x^o$.
Step 2 *Look for the ordered region,* O_{σ^o} *which* y^* *belong to, where* σ^o *determines the order.*
Step 3 *Solve the linear program* P_{σ^0}. *Let* $u^0 = (x_1^0, x_2^0, z_\sigma^0)$ *be an optimal solution. If* $x^0 = (x_1^0, x_2^0) \notin O_{\sigma^o}$ *then determine a new ordered region* O_{σ^o}, *where* x^0 *belongs to and go to Step 3.*
Step 4 *Let* $y^o = (x_1^0, x_2^0)$.
Step 5 *If* y^o *belongs to the interior of* O_{σ^o} *then set* $y^* = y^0$ *and go to Step 8.*
Step 6 *If* $F(y^o) \neq F(y^*)$ *then* $\mathcal{L} := \{\sigma^0\}$
Step 7 *If there exist i and j verifying*

$$\gamma(y^o - a_{\sigma_i^o}) = \gamma(y^o - a_{\sigma_j^o}) \quad i < j \quad \text{such that}$$

$$(\sigma_1^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_n^o) \notin \mathcal{L}$$

then do
a) $y^* := y^o$, $\sigma^o := (\sigma_1^o, \sigma_2^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_M^o)$
b) $\mathcal{L} := \mathcal{L} \cup \{\sigma^o\}$
c) go to Step 3
else go to Step 8 (Optimum found)
Step 8 *Output* y^*

The above algorithm is efficient in the sense that it is polynomially bounded. Once the dimension of the problem is fixed, its complexity is dominated by the complexity of solving a linear program for each ordered region. Since the number of ordered regions is polynomially bounded and the interior point method solves linear programs in polynomial time, Algorithm 3.1 is polynomial in the number of cells.

Example 3.1. Consider the problem

$$\min_{x \in \mathbb{R}^2} \gamma(x - A)_{(1)} + 2\gamma(x - A)_{(2)} + 3\gamma(x - A)_{(3)}$$

where $A = \{(3,0),(0,11),(16,8)\}$ and γ_B is the hexagonal polyhedral norm with $Ext(B) = \{(2,0),(1,2),(-1,2),(-2,0),(-1,-2),(1,-2)\}$.

We show in Figure 4 the generalized elementary convex sets for this problem. Notice that the thick lines represent the bisectors for the points in A, while the thin ones are the fundamental directions of the norm. We solve the problem using Algorithm 3.1. Starting with $x^o = (0,11)$ we get the optimal solution in two iterations. In the first one, we get the point $x^1 = (6.5, 8)$ with objective value 26.25. In the second iteration, we obtain $x^2 = (7,8)$ with objective value 26. This point cannot be improved in its neighborhood, therefore it is the optimal solution.

The iterations given by the algorithm for this example are depicted in Figure 4.

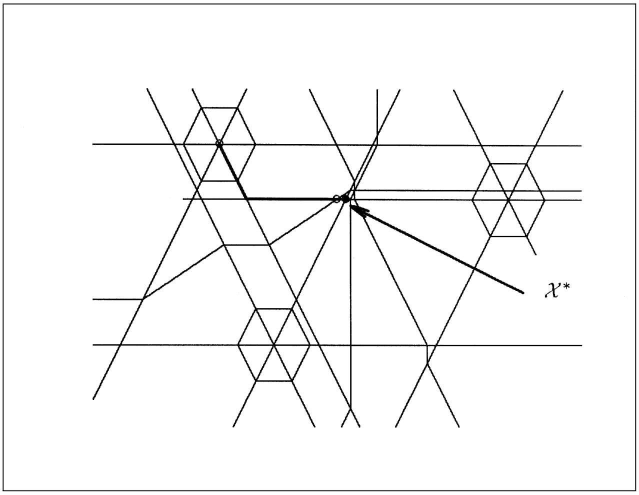

Fig. 4. Optimal solution

If we allow the lambdas to be non-monotone we loose the convexity of $F(x)$. Therefore the linear programming approach in Algorithm 3.1 becomes quite expensive, since we need to look at all possible ordered regions without explicitly constructing them. This means to search through an exponential number of regions. However, we can use the explicit construction of the ordered regions developed in the previous section. Now there are two ways to get an efficient algorithm even in the non-convex case. First, we can construct the ordered regions explicitly, solve the linear programs P_σ for all the $O(M^4G^4)$ ordered regions and determine the globally best solution. Secondly, we can construct all intersection points of all fundamental directions together with the all bisectors. This set constitutes a FDS (finite dominating set) for any ordered Weber problem in the plane, since the objective function behaves linear on the g.e.c.s and the described intersection points are just the extreme points of the g.e.c.s.

Summing up, we developed an efficient descent algorithm for the convex case and also two possibilities for efficient algorithms in the general case.

4 Extension to the multifacility case

A natural extension of the single facility model is to consider the location of N new facilities rather than only one. In this formulation the new facilities are chosen to provide service to all the existing facilities minimizing an ordered objective function. These ordered problems are of course harder to handle than the classical ones not considering ordered distances. Therefore, as no detailed complexity results are known for the ordinary multifacility problem nothing can be said about the complexity of the ordered Weber problem. Needless to say that its resolution is even much more difficult than for single facility models.

Before formalizing the above problem, we will distinguish two different approaches that come from two different interpretations of the new facilities to be located. The first one assumes that the new facilities are not interchangeable, which means that they are of different importance for each one of the existing facilities. The second one assigns the same importance to all new facilities. Here, we are only interested in the size of the distances, which means that we do not consider any order among the new facilities and look for equity in the service, minimizing the largest distances.

4.1 The non-interchangeable multifacility model

Let us consider a set of demand points $A = \{a_1, a_2, \ldots, a_M\}$. We want to locate N new facilities $X = \{x_1, x_2, \ldots, x_N\}$ which minimize the following expression:

$$F_I(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_{ij} \gamma(x_i - A)_{(j)} + \sum_{k=1}^{N} \sum_{l=1}^{N} \mu_{kl} \gamma(x_k - x_l) \quad (3)$$

where

$$\lambda_{11} \leq \lambda_{12} \leq \cdots \leq \lambda_{1M}$$

$$\lambda_{21} \leq \lambda_{22} \leq \cdots \leq \lambda_{2M}$$

$$\ldots$$

$$\lambda_{N1} \leq \lambda_{N2} \leq \cdots \leq \lambda_{NM};$$

$\mu_{kl} \geq 0$ for any $k = 1, \ldots, N$, $l = 1, \ldots, N$ and $\gamma(x_i - A)_{(j)}$ is the expression, which appears at the j-th position in the ordered list

$$\mathcal{M}_i = \{w_p \gamma(x_i - a_p), p = 1, 2, \ldots, M\} \quad \text{for } i = 1, 2, \ldots, N. \quad (4)$$

Note that in this formulation we assign the lambda parameters with respect to each new facility, i.e., x_j is considered to be non-interchangeable with x_i whenever $i \neq j$. For this reason we say that this model has non-interchangeable facilities. With the same classification scheme [16] used for the single facility model, we will refer to this problem as $N/\mathbb{R}^2/\lambda_{ord}/\gamma_B/\Sigma_{ord}$.

In order to illustrate this approach we show an example which will serve as motivation for the following:

Consider a distribution company with a central department and different divisions to distribute different commodities. A new set of final retailers appears in a new area. Each division wants to locate a new distribution center to supply the demand generated by these retailers with respect to its own strategic objective. The overall operation cost is the sum of the operation cost of the different divisions plus the administrative cost which only depends of the distances between the new distribution centers and which is supported by the central department. It is evident that for each division the quality of its distribution center only depends on the orders of the distances to the retailers but not on the name given to them, i.e. it depends on the ordered distances from each new facility to the retailers. Therefore, for this situation an ordered multifacility non-interchangeable model with a new facility for each commodity (division) of the company should be used.

As in the single facility model we can prove that the objective function (3) is convex, which eases the analysis of the problem and the development of an efficient algorithm.

Proposition 4.1. *The objective function F_I is convex.*

Proof: We know that

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\lambda_{ij}\gamma(x_i - A)_{(j)} = \sum_{i=1}^{N}\max_{\sigma^i}\sum_{j=1}^{N}\lambda_{ij}w_{\sigma^i_j}\gamma(x_i - a_{\sigma^i_j})$$

where σ^i is a permutation of the set $\{1, 2, \ldots, M\}$. Therefore, the first part of the objective function is a sum of maxima of convex functions analogous to Lemma 3.1. Hence, it is a convex function. On the other hand, the second term of the objective function F_I is convex. Thus, F_I is a convex function as a sum of convex functions. □

The problem $N/\mathbb{R}^2/\lambda_{ord}/\gamma_B/\sum_{ord}$ can be transformed within the new ordered regions in the same way that we did for $1/\mathbb{R}^2/\lambda_{ord}/\gamma_B/\sum_{ord}$. It should be noted that in $\mathbb{R}^2$ the subdivisions induced by the ordered regions of this problem are given as intersection of N subdivisions. Each one of these N subdivisions determines the ordered regions of each new facility.

Let $\sigma^k = (\sigma^k_1, \ldots, \sigma^k_M)$ $k = 1, \ldots, N$ be the permutations which give the order of the lists $\mathscr{M}_k$ introduced in (4). Consider the following linear program (P^I_σ):

$$\min \sum_{k=1}^{N}\sum_{l=1}^{M}\lambda_{kl}z_{k\sigma^k_l} + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_{ij}y_{ij}$$

s.t.

$$w_l\langle e^o_g, x_k - a_l\rangle \le z_{kl} \quad e^o_g \in B^o, \quad k = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, M$$

$$\langle e^o_g, x_i - x_j\rangle \le y_{ij} \quad i = 1, 2, \ldots, N, \quad j = i + 1, \ldots, M$$

$$z_{k\sigma^k_l} \le z_{k\sigma^k_{l+1}} \quad k = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, M - 1$$

Then, Algorithm 3.1 can easily be adapted to accommodate the multifacility case. Note than in contrary to that algorithm where we look for one point in $\mathbb{R}^2$ we now look for N points in $\mathbb{R}^2$ or equivalently for one point in $\mathbb{R}^{2N}$. To do that, we only have to modify Step 1 by choosing N starting points instead of one. In addition, we also have to consider that now the ordered regions are defined by different permutations, one from each list $\mathscr{M}_i$. Therefore, we have to replace the linear program P_σ by P^I_σ and to adapt its set of optimal solutions.

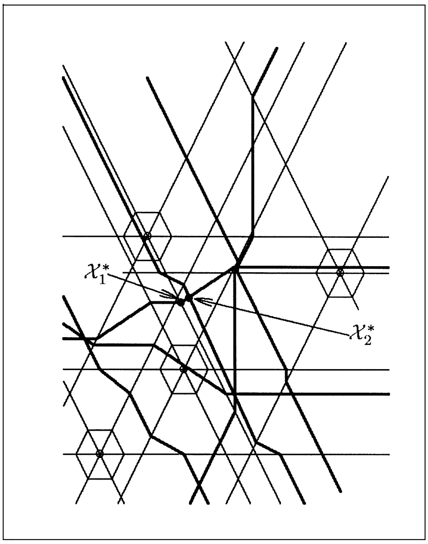

Fig. 5. Illustration for Example 4.1

Since this algorithm is essentially the same that the one proposed for the single facility model, we can conclude that it is also polynomial bounded, hence applicable.

Example 4.1. Consider the two-facility problem.

$$\min_{x_1, x_2 \in \mathbb{R}^2} 2.5\gamma_B(x_1 - A)_{(4)} + 2\gamma_B(x_1 - A)_{(3)} + 1.5\gamma_B(x_1 - A)_{(2)}$$

$$+ \gamma_B(x_1 - A)_{(1)} + 0.75\gamma_B(x_2 - A)_{(4)} + 0.1\gamma_B(x_2 - A)_{(3)}$$

$$+ 0.1\gamma_B(x_2 - A)_{(2)} + 0.1\gamma_B(x_2 - A)_{(1)} + 0.5\gamma_B(x_1 - x_2)$$

where $A = \{(3,0), (0,11), (16,8), (-4,-7)\}$, and γ_B is the hexagonal polyhedral norm, which we used in Example 3.1.

We obtain in the second iteration the optimal solution, with starting points $x_1^o = (0,11)$ and $x_2^o = (16,8)$. The optimal solution is $(2.75, 5.5)$ and $(3.125, 5.875)$. The elementary convex sets and the optimal solution can be seen in Figure 5.

4.2 The indistinguishable multifacility model

The multifacility model that we are considering now differs from the previous one in the sense that the new facilities are similar from the users point of view. Therefore, the new facilities have no different importance with respect to the existing ones. On the contrary, the weight given to each one of these new facilities depends only on the size of the distances.

Using the same notation as in Section 4.1, the objective function of this model is:

$$F_{II}(x_1, x_2, \dots, x_N) = \sum_{j=1}^{NM} \lambda_j \gamma(x - A)_{(j)} + \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_{ij} \gamma(x_i - x_j)$$

where

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{NM}$$

and $\gamma(x - A)_{(j)}$ is the expression which appears at the j-th position in the following ordered list

$$\mathscr{M}_{II} = \{w_p \gamma(x_k - a_p), k = 1, 2, \dots, N, p = 1, 2, \dots, M\}.$$

According to the classification scheme this problem is written as $N/\mathbb{R}^2/\bullet/\gamma_B/\sum_{ord}$. Also this model is motivated by a hypothetical real situation:

Consider the same multi-commodity distribution company as in the last subsection. However, assume now that in this case the strategic objective is not fixed by each division but it is fixed by the central department. Therefore, the objective is still to locate a distribution center for each commodity but the total cost of the distribution is supported by the central department. Thus, the complete order of the distances to the retailers from the distribution centers is important rather than the distances with respect to each one of the centers for the different commodities. With these hypotheses this situation can be formulated as a multifacility indistinguishable model where we want to locate as many facilities as different commodities in the company depending on complete the size of the distances.

Proposition 4.2. *The objective function F_{II} is convex.*

The proof is analogous to the one given for Proposition 4.1. □

Using again the same strategy that we have already used for the non-interchangeable multifacility model, the problem $N/\mathbb{R}^2/\bullet/\gamma_B/\sum_{ord}$ can be solved using an adaptation of Algorithm 3.1.

Let σ be a permutation of $\{1, \dots, MN\}$ where $\sigma_{(k-1)M+j}$ gives the position of $w_j \gamma(x_k - a_j)$ in $\mathscr{M}_{II}$.

Consider the following linear programming problem (P_σ^{II}):

$$\min \sum_{k=1}^{N} \sum_{l=1}^{M} \lambda_{\sigma_{kl}} z_{\sigma_{(k-1)M+l}} + \sum_{p=1}^{N} \sum_{q=1}^{N} \mu_{pq} y_{pq}$$

s.t.

$$w_l \langle e_g^o, x_k - a_l \rangle \le z_{(k-1)M+l} \quad e_g^o \in B^o \quad k = 1, 2, \dots, N \quad l = 1, 2, \dots, M$$

$$\langle e_g^o, x_p - x_q \rangle \le y_{pq} \quad p = 1, 2, \dots, N \quad q = 1, \dots, N$$

$$z_{\sigma_{(k-1)M+l}} \le z_{\sigma_{(k-1)+l+1}} \quad k = 1, 2, \dots, N \quad l = 1, 2, \dots, M - 1$$

Once we replace P_σ^I by P_σ^{II} we can easily adapt the algorithmic approach showed for the previous model in Section 4.1. Hence, the same conclusions that we obtained for $N/\mathbb{R}^2/\lambda_{ord}/\gamma_B/\sum_{ord}$ are applicable to $N/\mathbb{R}^2/\bullet/\gamma_B/\sum_{ord}$.

5 Extensions

5.1 Restricted case

In the last years an extension to classical location models which has attained considerable attention are the restricted facility location problems. When solving restricted location problems a forbidden region, where it is not allowed to place a new facility has to be respected (see for instance, Brady and Rosenthal [3], Drezner [7], Karkazis [14], Aneja and Palar [1]). Also the work of Francis et al. [9] in which a contour line approach is given is related to this topic and Hamacher and Nickel [10] and Nickel [16], describe a general discretization concept for solving restricted location problems.

In this section we will study the ordered Weber problems with forbidden regions.

We will assume that there is a forbidden region $\mathcal{R}$ containing all the optimal solutions of the unrestricted problem. This hypothesis is necessary because otherwise we can get the optimal solution by solving the unrestricted problem. Moreover, if the number of forbidden regions is greater than one, to obtain the optimal solution of the restricted problem, we only have to consider the forbidden region which contains the optimal solution of the unrestricted problem.

As basis of the solution procedure we need the following result.

Theorem 5.1. *For* $1/\mathbb{R}^2/\mathcal{R}/\gamma_B/\sum_{ord}$ *with polyhedral gauges there is always an optimal solution on the 0-dimensional intersections between the boundary of* $\mathcal{R}$*, the fundamental directions and the bisector lines.*

Proof: Using the same arguments as in Theorem 2.4.5 in [16], it follows that the optimal solutions of the restricted ordered facility location problem is on the boundary of the forbidden region. Moreover, the objective function is linear in each generalized elementary convex set, see Lemma 3.4, and the proof follows analogous to [16]. □

As an immediate consequence of the Theorem 5.1 we state the following algorithm for solving the single facility problems with a forbidden region, $\mathcal{R}$.

ALGORITHM 5.1.

Solving $1/\mathbb{R}^2/\mathcal{R}/\gamma_B/\sum_{ord}$

Step 1 *Compute the fundamental directions and bisector lines for all existing facilities.*

Step 2 *Determine* $\{y_1, y_2, \ldots, y_k\}$ *the intersection points of fundamental directions or bisector lines and the boundary of the forbidden region,* $\mathcal{R}$.

Step 3 *Compute* $x^*_{\mathcal{R}} \in \arg\min\{f(y_1), f(y_2), \ldots, f(y_k)\}$ *(*$x^*_{\mathcal{R}}$ *is an optimal solution to the restricted location problem).*

Step 4 *The set of optimal solutions is* $\{x : f(x) = f(x^*_{\mathscr{R}})\}$ *intersected with the boundary of* $\mathscr{R}$.

For the particular case of polyhedral forbidden regions we can get better results. Let $\mathscr{R}$ be a polyhedral forbidden region, $\{s_1, s_2, \ldots, s_k\}$ the set of facets of $\mathscr{R}$ and $A = \{a_1, \ldots, a_M\}$ the set of existing facilities.

ALGORITHM 5.2.

<u>*Solving* $1/\mathbb{R}^2/\mathscr{R} = $ *convex polyhedron*$/\gamma_B/\sum_{ord}$</u>

Step 1 *Let* $p := 1$, $\mathscr{L} := \emptyset$ *and let* y^* *be an arbitrary feasible solution.*
Step 2 *Consider the hyperplane* $\mathscr{T}_p$ *defined by the facet* s_p *of* $\mathscr{R}$ *and choose* x^o *belonging to the relative interior of* s_p. *Let* $\mathscr{T}_p^{\leq}$ *be the halfplane which does not contain* $\mathscr{R}$ *and let* $x^* = x^o$.
Step 3 *Determine the ordered region* O_{σ^o} *where* x^o *belongs to, and the permutation* σ^o *which determines this region.*
Step 4 *Solve the following linear program*

$$\begin{aligned} \min \quad & \sum_{i=1}^{M} \lambda_i z_{\sigma_i^o} \\ s.t. \quad & w_i \langle b_g, x - a_i \rangle \leq z_i \quad e_g \in B^o, \quad i = 1, 2, \ldots, M, \\ & z_{\sigma_i^o} \leq z_{\sigma_{i+1}^o} \qquad i = 1, 2, \ldots, M-1, \\ & x \in \mathscr{T}_p^{\leq}. \end{aligned} \qquad (P_{\mathscr{T}^{\leq}})$$

Step 5 *Let* $u^o = (x^o, z_\sigma^o)$ *be an optimal solution of* $P_{\mathscr{T}_p^{\leq}}$. *If* $x^o \notin O_{\sigma^o}$ *then go to Step 3.*
Step 6 *If* x^0 *belongs to the interior of* O_{σ^o} *then let* $x^* = x^o$ *and go to Step 9*
Step 7 *If* $F(x^o) \neq F(x^*)$ *then* $\mathscr{L} := \emptyset$
Step 8 *If there exist i and j verifying*

$$\gamma(x^o - a_{\sigma_j^o}) = \gamma(x^o - a_{\sigma_i^o}) \quad i < j \quad \text{such that}$$

$$(\sigma_1^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_M^o) \notin \mathscr{L}$$

Then do

a) $x^* := x^o$, $\sigma^o := (\sigma_1^o, \sigma_2^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_M^o)$
b) $\mathscr{L} := \mathscr{L} \cup \{\sigma^o\}$
c) *GO TO Step 4.*

Step 9 *Do*
a) *If* $F(x^*) < F(y^*)$ *then* $y^* := x^*$
b) $p := p + 1$.
d) *If* $p < k$ *GO TO Step 2, otherwise the optimal solution is* y^*.

Notice that this algorithm can be used to solve problems with convex forbidden regions not necessarily polyhedral. In order to do so we only have to approximate these regions by polyhedral ones. Since this approximation can be done with arbitrary precision using for instance the sandwich approximation in [4] and [13], we can get good approximations to the optimal solutions of the original problems.

5.2 Non polyhedral case

In the previous sections we considered polyhedral gauges. For some special cases the results presented so far can be adapted also to non polyhedral gauges. In the case of the Euclidean norm, for example, numerous algorithms exist to determine the bisectors needed for constructing the ordered regions (see [17]). Then, instead of linear programs, convex programs have to be solved in these ordered regions. However, this is not possible for general gauges and therefore will use the results of the polyhedral case to develop a general scheme for solving the considered problems under general gauges (non necessarily polyhedral).

We show that the optimal solutions of these problems can be arbitrarily approximated by sequence of optimal solutions of problems with polyhedral gauges converging under the Hausdorff metric to the considered non-polyhedral one.

Although in this section, we only consider the objective function of the single facility case, $F(x)$, all the results can be extended in an easy way to the multifacility cases.

Let B be a unit ball of the gauge $\gamma_B(\cdot)$, $\{B_n\}_{n\in\mathbb{N}}$ an increasing sequence of polyhedra included in B and $\{B^n\}_{n\in\mathbb{N}}$ a decreasing sequence of polyhedra including B, that is,

$$B_n \subset B_{n+1} \subset B \subset B^{n+1} \subset B^n \quad \text{for all } n = 1, 2, \ldots$$

Let $\gamma_{B_n}(\cdot)$ and $\gamma_{B^n}(\cdot)$ be the gauges whose unit balls are B_n and B^n respectively.

Proposition 5.1. *If* $B_n \subset B \subset B^n$ we have that

$$\gamma_{B_n}(x) \geq \gamma_B(x) \geq \gamma_{B^n}(x) \quad \forall x \in \mathbb{R}^2$$

The proof follows directly from the definition of gauges.

Recall that given two compact sets A, B the Hausdorff distance between A and B is

$$d_H(A, B) = \max\left(\max_{x\in A} d_2(x, B), \max_{y\in B} d_2(A, y) \right)$$

where $d_2(x, B) = \min_{y\in B} d_2(x, y)$ being d_2 the Euclidean distance.

Proposition 5.2. *Let K be a compact set. If B_n converges to B and B^n converges to B under the Hausdorff metric then for all* $\varepsilon > 0$ *there exists n_0 such that for all $n \geq n_0$*

$$\max_{x \in K} |F_n(x) - F(x)| < \varepsilon$$

$$\max_{x \in K} |F^n(x) - F(x)| < \varepsilon$$

being $F_n(x) := \sum_{i=1}^{M} \lambda_i \gamma_{B_n}(x - A)_{(i)}$ *and* $F^n(x) := \sum_{i=1}^{M} \lambda_i \gamma_{B^n}(x - A)_{(i)}$.

Proof: We only prove the first inequality. The second one follows analogously.

Since B_n converges to B under the Hausdorff metric verifying $B_n \subset B_{n+1}$ for all n, and K is a compact set then given $\varepsilon > 0$ there exists n_a for all $a \in A$ such that if $n > n_A := \max_{a \in A} n_a$ then $|\gamma_B(x - a) - \gamma_{B_n}(x - a)| < \dfrac{\varepsilon}{\sum_{i=1}^{M} w_i \sum_{i=1}^{M} \lambda_i}$ $\forall x \in K$.

By continuity we have that for any i, j and any $x \in K$ verifying that $w_i \gamma_B(x - a_i) < w_j \gamma_B(x - a_j)$ there exists n_0 such that for all $n > n_0$

$$w_i \gamma_{B_n}(x - a_i) < w_j \gamma_{B_n}(x - a_j)$$

On the other hand, if there exists k, l and $x \in K$ such that $w_k \gamma_B(x - a_k) = w_l \gamma_B(x - a_l)$ then there also exists n_0 and a permutation σ^{n_0} such that for all $n > n_0$ it holds: 1) $w_{\sigma_k^{n_0}} \gamma_{B_n}(x - a_{\sigma_k^{n_0}}) = \gamma_{B_n}(x - A)_{(k)}$, and 2) $w_{\sigma_k^{n_0}} \gamma_B(x - a_{\sigma_k^{n_0}}) = \gamma_B(x - A)_{(k)}$. Hence, we have for any $x \in K$ and $n > \max\{n_A, n_0\}$ that

$$\gamma_B(x - A)_{(k)} = w_{\sigma_k^{n_0}} \gamma_B(x - a_{\sigma_k^{n_0}})$$

$$\gamma_{B_n}(x - A)_{(k)} = w_{\sigma_k^{n_0}} \gamma_{B_n}(x - a_{\sigma_k^{n_0}}).$$

Therefore for any $x \in K$ and $n > \max\{n_A, n_0\}$ we obtain that

$$|F_n(x) - F(x)| = \sum_{i=1}^{M} \lambda_i |\gamma_B(x - A)_{(i)} - \gamma_{B_n}(x - A)_{(i)}|$$

$$= \sum_{i=1}^{M} \lambda_i w_{\sigma_i^{n_0}} |\gamma_B(x - a_{\sigma_i^{n_0}}) - \gamma_{B_n}(x - a_{\sigma_i^{n_0}})| < \varepsilon. \qquad \square$$

Corollary 5.1. *i) If B_n converges to B under the Hausdorff metric, then $F_n(x)$ converges to $F(x)$, and the sequence $\{F_n(x)\}_{n \in \mathbb{N}}$ is decreasing.*
ii) If B^n converges to B under the Hausdorff metric, then $F^n(x)$ converges to $F(x)$, and the sequence $\{F^n(x)\}_{n \in \mathbb{N}}$ is increasing.

In the following, we use another kind of convergence, called epi-convergence see Definition 1.9 in the book of Attouch [2]. Let $\{g; g^v, v = 1, 2, \ldots\}$ be a collection of extended-values functions. We say that g^v epi-converges to g if for all x,

$$\inf_{x^v \to x} \liminf_{v \to \infty} g^v(x^v) \geq g(x)$$

$$\inf_{x^v \to x} \limsup_{v \to \infty} g^v(x^v) \leq g(x)$$

where the infima are with respect to all subsequences converging to x. The epi-convergence is very important because it establishes a relationship between the convergence of functionals and the convergence of the sequence of their minima. Further details can be found in the book of Attouch [2].

Our next result states the theoretical convergence of the proposed scheme.

Theorem 5.2. *i) Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $x_n \in \arg\min F_n(x)$ *then any accumulation point of* $\{x_n\}_{n\in\mathbb{N}}$ *belongs to* $\arg\min F$.

ii) Let $\{x^n\}_{n\in\mathbb{N}}$ *be a sequence such that* $x^n \in \arg\min F^n(x)$ *then any accumulation point of* $\{x_n\}_{n\in\mathbb{N}}$ *belongs to* $\arg\min F$.

Proof: We only prove the first part, because the proof of the second one is built on the same pattern using Proposition 2.41. in [2] instead of Proposition 2.48.

First of all, since the sequence $\{F_n\}_{n\in\mathbb{N}}$ is a decreasing sequence applying Theorem 2.46 in [2] we obtain that the sequence $\{F_n(x)\}_{n\in\mathbb{N}}$ is epi-convergent.

In addition, we get from Proposition 2.48 in [2] that

$$\lim_{n\to\infty} \inf_{x\in\mathbb{R}^2} F_n(x) = \inf_{x\in\mathbb{R}^2} \lim_{n\to\infty} F_n(x) = \inf_{x\in\mathbb{R}^2} F(x) \tag{5}$$

Since $\mathbb{R}^2$ is a first countable space and $\{F_n\}_{n\in\mathbb{N}}$ is epi-convergent,we get from Theorem 2.12 in [2] that any accumulation point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an optimal solution of the problem with objective function F. □

5.3 Conclusions

In this paper we have presented efficient algorithms for the ordered Weber problem introduced by [19] for the case of polyhedral gauges. Even for the case of a non-convex objective function a polynomial global optimization algorithm could be derived by using geometric properties of the problem. Also extensions to the multifacility case have been developed. In addition a discussion of the non polyhedral case and the case with forbidden regions has been presented. Therefore, we have provided a new flexible tool for modeling and solving a broad range of location problems.

As already mentioned in the introduction most of the results can easily be extended to $\mathbb{R}^n$.

Further research includes the analysis of a multi-criteria formulation of these problems as well as a detailed study about ordered Weber problems with some negative weights. Also an analogous model for network location problems is under research.

Acknowledgments: The research of the first, thrid and fourth authors is partially supported by Spanish Ministerio de Educación y Cultura grants numbers HA1997-0123 and PB97-0707. The authors would also like to thank an anonymous referee who helped to improve the paper significantly.

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